Chapter review 2

1 $3x^2 - 4x + 6 = 0$ has roots α and β
$a = 3,\ b = -4$ and $c = 6$

a i $\alpha + \beta = -\frac{b}{a}$
$= \frac{4}{3}$

ii $\alpha\beta = \frac{c}{a}$
$= \frac{6}{3}$
$= 2$

b $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - (3\alpha^2\beta + 3\alpha\beta^2)$
$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$= \left(\frac{4}{3}\right)^3 - 3(2)\left(\frac{4}{3}\right)$
$= -\frac{152}{27}$ as required

c The sum of the roots is
$\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2}$
$= \frac{-\frac{152}{27}}{(\alpha\beta)^2}$ using result from **b**
$= \frac{-152}{27(2)^2}$
$= -\frac{38}{27}$

The product of the roots is
$\frac{\alpha}{\beta^2}\left(\frac{\beta}{\alpha^2}\right) = \frac{\alpha\beta}{\alpha^2\beta^2}$
$= \frac{1}{\alpha\beta}$
$= \frac{1}{2}$

So $x^2 + \frac{38}{27}x + \frac{1}{2} = 0$
$54x^2 + 76x + 27 = 0$

2 $\alpha + \beta = -\frac{5}{2}$ and $\alpha\beta = -6$

a $x^2 + \frac{5}{2}x - 6 = 0$
$2x^2 + 5x - 12 = 0$

b i $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= \left(-\frac{5}{2}\right)^2 - 2(-6)$
$= \frac{73}{4}$

ii $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - (3\alpha^2\beta + 3\alpha\beta^2)$
$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$= \left(-\frac{5}{2}\right)^3 - 3(-6)\left(-\frac{5}{2}\right)$
$= -\frac{485}{8}$

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2 c The sum of the roots is

$$\alpha-\frac{1}{\alpha^2}+\beta-\frac{1}{\beta^2}=\alpha+\beta-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)$$

$$=\alpha+\beta-\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$

$$=-\frac{5}{2}-\frac{\frac{73}{4}}{(-6)^2}$$

$$=-\frac{433}{144}$$

The product of the roots is

$$\left(\alpha-\frac{1}{\alpha^2}\right)\left(\beta-\frac{1}{\beta^2}\right)=\left(\frac{\alpha^3-1}{\alpha^2}\right)\left(\frac{\beta^3-1}{\beta^2}\right)$$

$$=\frac{\alpha^3\beta^3-\alpha^3-\beta^3+1}{\alpha^2\beta^2}$$

$$=\frac{(\alpha\beta)^3-\left(\alpha^3+\beta^3\right)+1}{(\alpha\beta)^2}$$

$$=\frac{(-6)^3-\left(-\frac{485}{8}\right)+1}{(-6)^2}$$

$$=-\frac{1235}{288}$$

So $x^2+\frac{433}{144}x-\frac{1235}{288}=0$

$288x^2+866x-1235=0$

3 $\alpha+\beta=-\frac{7}{3}$ and $\alpha\beta=-2$

a $x^2+\frac{7}{3}x-2=0$

$3x^2+7x-6=0$

3 b $3x^2+7x-6=0$

$\Rightarrow(3x-2)(x+3)=0$

$x=\frac{2}{3}$ or $x=-3$

α and β are roots of the equation

$\alpha>\beta$ so $\alpha=\frac{2}{3}$ and $\beta=-3$

Therefore

$$\alpha-\beta=\frac{2}{3}-(-3)$$

$$=\frac{11}{3} \text{ as required}$$

c The sum of the roots is

$$\frac{\alpha+\beta}{\alpha}+\frac{\alpha-\beta}{\beta}=\frac{\beta(\alpha+\beta)+\alpha(\alpha-\beta)}{\alpha\beta}$$

$$=\frac{\alpha\beta+\beta^2+\alpha^2-\alpha\beta}{\alpha\beta}$$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\left(-\frac{7}{3}\right)^2-2(-2)}{(-2)}$$

$$=-\frac{85}{18}$$

The product of the roots is

$$\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{\alpha-\beta}{\beta}\right)=\frac{(\alpha+\beta)(\alpha-\beta)}{\alpha\beta}$$

$$=\frac{\left(-\frac{7}{3}\right)\left(\frac{11}{3}\right)}{(-2)}$$

$$=\frac{77}{18}$$

So $x^2+\frac{85}{18}x+\frac{77}{18}=0$

$18x^2+85x+77=0$

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Challenge

1 $2x^2+px+q=0$ has roots α and β

$16x^2+57x+16=0$ has roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

a i From the first equation

$\alpha+\beta=-\frac{p}{2}$ and $\alpha\beta=\frac{q}{2}$

From the second equation

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-\frac{57}{16}$$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\left(-\frac{p}{2}\right)^2-2\left(\frac{q}{2}\right)}{\frac{q}{2}}$$

$$=\frac{2\left(-\frac{p}{2}\right)^2-4\left(\frac{q}{2}\right)}{q}$$

Since $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-\frac{57}{16}$

$$\frac{2\left(-\frac{p}{2}\right)^2-4\left(\frac{q}{2}\right)}{q}=-\frac{57}{16}$$

$$32\left(-\frac{p}{2}\right)^2-64\left(\frac{q}{2}\right)=-57q$$

$8p^2-32q=-57q$

$8p^2+25q=0$

Since $p+q=-3$

$q=-3-p$

$8p^2+25(-p-3)=0$

$8p^2-25p-75=0$

$(8p+15)(p-5)=0$

$p=-\frac{15}{8}$ or $p=5$

Since p is a positive integer, $p=5$

ii $q=-3-p$

$q=-8$

b $2x^2+5x-8=0$

$$x=\frac{-5\pm\sqrt{5^2-4(2)(-8)}}{2(2)}$$

$$=\frac{-5\pm\sqrt{89}}{4}$$

α and β are the roots of the equation

$\alpha>\beta$

$\therefore \alpha=\frac{-5+\sqrt{89}}{4}$ and $\beta=\frac{-5-\sqrt{89}}{4}$

Therefore

$$\alpha-\beta=\frac{-5+\sqrt{89}}{4}-\left(\frac{-5-\sqrt{89}}{4}\right)$$

$$=\frac{-5+\sqrt{89}}{4}+\frac{5+\sqrt{89}}{4}$$

$$=\frac{\sqrt{89}}{2}$$

2 $4x^2+px+q=0$ has roots α and α^*

a $\alpha+\alpha^*=-\frac{p}{4}$ and $\alpha\alpha^*=\frac{q}{4}$

Since Re $(\alpha)=-3$

$\alpha=-3+bi$ and $\alpha^*=-3-bi$

$\alpha+\alpha^*=(-3+bi)+(-3-bi)$

$=-6$

and since

$\alpha+\alpha^*=-\frac{p}{4}$

$-\frac{p}{4}=-6$

$p=24$

b $\alpha=-3+bi$ and $\alpha^*=-3-bi$

$\alpha\alpha^*=(-3+bi)(-3-bi)$

$=9+b^2$

and since $\alpha\alpha^*=\frac{q}{4}$

$9+b^2=\frac{q}{4}$

$q=36+4b^2$

Since Im $(\alpha)\neq 0$, $b^2>0$

Therefore $q>36$

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